Note on Doron Zeilberger's paper $\binom{5}{2}$ *proofs that* $\binom{n}{k} \leq \binom{n}{k+1}$ *if* $k < n/2$

Murali K. Srinivasan
Department of Mathematics
Indian Institute of Technology, Bombay
Powai, Mumbai 400076, INDIA
mks@math.iitb.ac.in,murali.k.srinivasan@gmail.com
Mathematics Subject Classifications: 05A15, 05A30.

To the memory of Mobi

In his very interesting paper [**Z**], Doron Zeilberger recalls, among other results, several proofs of unimodality of the binomial coefficients. Among the combinatorial proofs he calls the beautiful symmetric chain decomposition (SCD) proof of de Bruijn, Tengbergen, and Kruyswijk [**A, BTK**] his personal favourite. He then goes on to present a variation of Proctor's [**P**] beautiful algebraic proof (based on the representation theory of the Lie algebra $\mathrm{sl}(2, \mathbb{C})$) of injectivity of the up operator on the lower half of the Boolean algebra (which implies unimodality), calling this the longest and yet the best proof.

The purpose of this note is to present yet another algebraic proof of the statement in the title: we show that the SCD proof has a simple and natural linear analog that proves injectivity of the up operator on the lower half (and surjectivity on the upper half).

For a finite set S, let $V(S)$ denote the complex vector space with S as basis. Let $B(n)$ denote the set of all subsets of $[n] = \{1, 2, \ldots, n\}$ and, for $0 \leq k \leq n$, let $B(n)_k$ denote the set of all k-subsets of $[n]$. Then we have $V(B(n)) = V(B(n)_0) \oplus V(B(n)_1) \oplus \cdots \oplus V(B(n)_n)$ (vector space direct sum). An element $v \in V(B(n))$ is *homogeneous* if $v \in V(B(n)_k)$ for some k. We say that a homogeneous element v is of *rank* k, and we write $r(v) = k$, if $v \in V(B(n)_k)$. The *up operator* $U : V(B(n)) \to V(B(n))$ is defined, for $X \in B(n)$, by $U(X) = \sum_Y Y$, where the sum is over all Y covering X, i.e., $X \subseteq Y$ and $|Y| = |X| + 1$. A *symmetric Jordan chain* (SJC) in $V(B(n))$ is a sequence $v = (v_1, \ldots, v_h)$ of nonzero homogeneous elements of $V(B(n))$ such that $U(v_{i-1}) = v_i$, for $i = 2, \ldots h$, $U(v_h) = 0$, and $r(v_1) + r(v_h) = n$, if $h \geq 2$, or else $2r(v_1) = n$, if $h = 1$. Note that the elements of the sequence v are linearly independent, being nonzero and of different ranks. A *symmetric Jordan basis* (SJB) of $V(B(n))$ is a basis of $V(B(n))$ consisting of a disjoint union of SJC's in $V(B(n))$.

Theorem *There exists an SJB of* $V(B(n))$.

Proof We give a constructive proof that inductively produces an explicit SJB of $V(B(n))$, the case $n = 0$ being clear.

Let $V = V(B(n+1))$. Define $V(0)$ to be the subspace of V generated by all subsets of $[n+1]$ not containing $n+1$ and define $V(1)$ to be the subspace of V generated by all subsets of $[n+1]$ containing $n+1$. We have $V = V(0) \oplus V(1)$. The linear map $R : V(0) \to V(1)$, given by $X \mapsto X \cup \{n+1\}$, $X \subseteq [n]$ is an isomorphism. We write $R(v) = \overline{v}$. Let U and U_0 denote, respectively, the up operators on V and $V(0)$ $(= V(B(n)))$. We have, for $v \in V(0)$,

$$U(v) = U_0(v) + \overline{v}, \;\; U(\overline{v}) = \overline{U_0(v)}. \tag{1}$$

By induction hypothesis there is an SJB $\mathcal{B}$ of $V(B(n)) = V(0)$. We shall now produce an SJB $\mathcal{B}'$ of V by producing, for each SJC in $\mathcal{B}$, either one or two SJC's in V such that the collection of all these SJC's is a basis.

Consider an SJC $(x_k, \ldots, x_{n-k})$ (for some $0 \leq k \leq \lfloor n/2 \rfloor$), where $r(x_k) = k$, in $\mathcal{B}$.

We now consider two cases.

(a) $k = n-k$: From (1) we have $U(x_k) = \overline{x_k}$ and $U(\overline{x_k}) = \overline{U_0(x_k)} = 0$. Since R is an isomorphism $\overline{x_k} \neq 0$. Add to $\mathcal{B}'$ the SJC

$$(x_k, \overline{x_k}). \tag{2}$$

(b) $k < n-k$: Set $x_{k-1} = x_{n+1-k} = 0$ and define

$$(y_k, \ldots, y_{n+1-k}), \text{ and } (z_{k+1}, \ldots, z_{n-k}), \tag{3}$$

by

$$y_l \;=\; x_l + (l-k)\,\overline{x_{l-1}}, \;\; k \leq l \leq n+1-k. \tag{4}$$
$$z_l \;=\; (n-k-l+1)\,\overline{x_{l-1}} - x_l, \;\; k+1 \leq l \leq n-k. \tag{5}$$

From (1) we have

$$U(\overline{x_l}) = \overline{U_0(x_l)} = \overline{x_{l+1}}, \; k \leq l \leq n-k \tag{6}$$

It thus follows from (1) and (6) that, for $k \leq l < n+1-k$, we have

$$U(y_l) = U(x_l + (l-k)\overline{x_{l-1}}) = x_{l+1} + \overline{x_l} + (l-k)\overline{x_l} = x_{l+1} + (l-k+1)\overline{x_l} = y_{l+1}.$$

Note that when $l = k$ the second step above is justified because of the presence of the $(l-k)$ factor even though $U(\overline{x_{k-1}}) = 0 \neq \overline{x_k}$. We also have $U(y_{n+1-k}) = U((n+1)\overline{x_{n-k}}) = (n+1)\overline{U_0(x_{n-k})} = 0$.

Similarly, for $k+1 \leq l < n-k$, we have

$$U(z_l) = U((n-k-l+1)\overline{x_{l-1}} - x_l) = (n-k-l+1)\overline{x_l} - x_{l+1} - \overline{x_l} = (n-k-l)\overline{x_l} - x_{l+1} = z_{l+1}.$$

and $U(z_{n-k}) = U(\overline{x_{n-k-1}} - x_{n-k}) = \overline{x_{n-k}} - \overline{x_{n-k}} = 0$.

Since $y_k = x_k \neq 0$, $y_{n+1-k} = (n+1)\overline{x_{n-k}} \neq 0$, x_l and $\overline{x_{l-1}}$ are linearly independent, for $k+1 \leq l \leq n-k$ and the 2×2 matrix

$$\begin{pmatrix} 1 & l-k \\ -1 & n-k-l+1 \end{pmatrix}$$

is nonsingular for $k+1 \leq l \leq n-k$, it follows that (3) gives two independent SJC's in V. Add these two to $\mathcal{B}'$.

Since $V = V(0) \oplus V(1)$ and R is an isomorphism it follows that performing the above step for each SJC in $\mathcal{B}$ we get an SJB $\mathcal{B}'$ of V. $\Box$

The main idea in the proof of the Theorem above has several consequences, studied in [**Sr**]. In that paper we show the following:

- Introduce the standard inner product on $V(B(n))$, i.e., the set $\{X \mid X \in B(n)\}$ is an orthonormal basis. It is shown that the SJB produced above is orthogonal. Moreover, any two SJC's starting at rank k and ending at rank $n-k$ look alike in the sense that the ratios of the lengths of the successive vectors is the same in both the chains and these ratios (i.e., the singular values) can be explicitly written down. This yields a new constructive proof of the explicit block diagonalization of the Terwilliger algebra of the binary Hamming scheme, recently achieved by Schrijver [**Sc**].
- It is shown that the SJB produced above is the canonically defined (upto common scalars on each SJC) symmetric Gelfand-Tsetlin basis of $V(B(n))$. This gives a natural representation-theoretic explanation for the orthogonality in the item above.
- The algorithm given above can be generalized to produce an SJB in the multiset case. The author's believes that the multiset version, which has a deeper level of recursion than the set case, deserves further study from a representation-theoretic viewpoint.

Acknowledgement

The author is grateful to Professor Doron Zeilberger whose encouraging e-mail motivated him to write this note.

References

[A] I. Anderson, Combinatorics of finite sets, *Clarendon Press, Oxford*, 1987.

[BTK] N. G. de Bruijn, C. A. v. E. Tengbergen and D. Kruyswijk, *On the set of divisors of a number*, Nieuw Arch. Wiskunde, 23: 191-193 (1951).

[P] R. A. Proctor, *Representations of $sl(2,\mathbb{C})$ on posets and the Sperner property*, SIAM Journal of Algebraic and Discrete Methods, 3:275-280 (1982).

[Sc] A. Schrijver, *New code upper bounds from the Terwilliger algebra and semidefinite programming*, IEEE Transactions on Information Theory, 51: 2859-2866 (2005).

[Sr] M. K. Srinivasan, *Symmetric chains, Gelfand-Tsetlin chains, and the Terwilliger algebra of the binary Hamming scheme*, arXiv. 1001.0280.

[Z] D. Zeilberger, $\binom{5}{2}$ *proofs that* $\binom{n}{k} \leq \binom{n}{k+1}$, *if* $k < n/2$, arXiv. 1003.1273